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CLASSIFICATION OF PRINCIPAL BUNDLES AND LIE GROUPOIDS WITH PRESCRIBED GAUGE GROUP BUNDLE*

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By the gauge group bundle of a principal bundle $P(B,G)$ we mean the Lie group bundle associated to $P(B,G)$ through the conjugacy action of G on itself. Given only B and a Lie group bundle M on B, we ask if there exists $P(B,G)$ with gauge group bundle isomorphic to M and, if so, how they can be described. Using a form of Whitehead's concept of crossed module, in place of the idea of an 'abstract kernel', we find an obstruction class in $\check{H}^2(B,ZG)$ (G the fibre-type of M) whose vanishing gives a necessary and sufficient condition for the existence of such a $P(B,G)$; and, when this class vanishes, a simple transitive action of $\check{H}^1(B,ZG)$ on the set of equivalence classes of possible bundles. We work mainly in terms of Lie groupoids, which language seems well-adapted to these questions.

Introduction

The classification of principal bundles $P(B,G)$ with prescribed base B and prescribed structure group G by nonabelian Čech cohomology $\check{H}^1(B,G)$ is very well known. However, although the concept of transition function, by which this classification is effected, is of great utility, it is less clear that the actual formulation in terms of Čech cohomology has been so useful — except in those cases (such as that of complex line bundles) where the group is actually abelian. Rather, this classification has provided one motivation for the study of nonabelian cohomology.

Here we formulate the classification problem differently and obtain a description which is entirely in terms of abelian Čech cohomology and, also, follows the pattern long-established by the extension theory of discrete groups, Lie algebras, and other algebraic structures. The key difference is that we prescribe not merely the structure group G, but the whole gauge transformation group bundle

$$\frac{P\times G}{G},$$

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also known as Ad P, the Lie group bundle whose module of sections is the gauge group. Thus we suppose given a manifold B and a Lie group bundle M on B and we seek principal bundles, $P(B,G)$ for which

$$\frac{P\times G}{G}\cong M,$$

as Lie group bundles over B. Even at this stage it is clear that such a $P(B,G)$ need not exist: if G is abelian, then

$$\frac{P\times G}{G}$$

is always trivializable so if M has abelian fibres but is non trivializable, then it cannot be a gauge group bundle.

We eliminate this type of counter example by requiring that B and M have, further, the structure of an 'abstract kernel' (in the terminology of [15]) in a suitably smooth sense. This is achieved by using a special case of Whitehead's concept of crossed module, which we call a coupling. This formulation enables us to bypass the problem that the inner automorphism group of a Lie group need not be closed in the full automorphism group.

The net result of these changes is a classification which is similar to, but distinct from, the classification of lifts of a principal bundle [9, 10, 24]: see Remark 4.1.

This approach to the classification of principal bundles developed out of work on Lie groupoids [17] and is most naturally expressed in terms of them. The concept of Lie groupoid is essentially equivalent to that of principal bundle, but avoids the arbitrary choice of base-point which often has to be made when a principal bundle is used to describe a geometric situation. For example, the Lie groupoid corresponding to the full frame bundle of a manifold B is the set of all isomorphisms between the various tangent spaces to B, rather than isomorphisms from $\mathbb{R}^n$ or a specified fibre; the Lie groupoid corresponding to the universal covering bundle $\tilde{B}(B,\pi_1 B)$ of a connected manifold B is the set of homotopy classes of all paths in B, rather than those with a prescribed end-point.

As a result of this impartiality, a Lie groupoid has no naturally distinguished structure group, but rather a bundle of groups; for the Lie groupoid corresponding to a principal bundle $P(B,G)$, this bundle is (isomorphic to)

$$\frac{P\times G}{G}.$$

Thus for a Lie groupoid it is natural to prescribe the whole gauge group bundle.

Section 1 is purely algebraic and relates the definition of coupling used here to the standard treatment of group cohomology. Section 2 briefly describes the Lie groupoid language. The main results are in Section 3. The last section contains a few remarks and examples, including a restatement of the main results in terms of principal bundles.

1. Crossed modules and couplings of discrete groups

Consider a group G and an abelian group A. An extension

$$A \rightarrowtail H \xrightarrow{\pi} G \tag{1}$$

of G by A induces a representation $\varrho : G \to \mathrm{Aut}(A)$ by $\varrho(g) = I_h|_A$, the restriction to A of the inner automorphism I_h of H corresponding to any h with $\pi(h) = g$. If one now replaces A by an arbitrary group N and considers an extension

$$N \rightarrowtail H \xrightarrow{\pi} G, \tag{2}$$

the automorphism $I_h|_N$ no longer depends solely on $\pi(h)$, and this process does not give a well-defined morphism $G \to \mathrm{Aut}(N)$. The usual way around this problem has been to consider the map

$$\varrho : G \to \mathrm{Out}(N) = \frac{\mathrm{Aut}(N)}{\mathrm{Inn}(N)},$$

$$g \mapsto \langle I_h|_N \rangle \quad \text{where } \pi(h) = g,$$

where $\mathrm{Inn}(N)$ is the group of inner automorphisms of N and $\langle \varphi \rangle$ denotes the $\mathrm{Inn}(N)$ coset of $\varphi \in \mathrm{Aut}(N)$. This ϱ is a well-defined morphism, called the *abstract kernel* of (2), and there is a standard classification of extensions (2) with a prescribed abstract kernel (see, for example, [15]).

Now if one wishes to carry this over to topological or Lie groups, one immediately encounters the problem that $\mathrm{Inn}(N)$ need not be closed in $\mathrm{Aut}(N)$. For Lie groups, especially, this is a real difficulty, for although theories of non-Hausdorff manifolds exist, one still wishes to characterize the smoothness of those morphisms $G \to \mathrm{Out}(N)$ which arise from extensions (2) of Hausdorff Lie groups.

A method for avoiding this problem, provided that G is connected, was given by Macauley [14]. However, the method we give here applies to all topological groups, requires no passage to universal covers, and is consonant with the new interpretations of higher-order cohomology introduced in the 1970's (for which see the references in [16]). Namely we replace the concept of abstract kernel by a particular type of crossed module, a concept introduced by Whitehead [23] in the context of homotopy theory. In effect, we replace the morphism $G \to \mathrm{Out}(N)$ by the pullback C in

$$\begin{array}{ccc} C & \longrightarrow & \mathrm{Aut}(N) \\ \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{Out}(N) \end{array}$$

together with a natural action of C on N.

Definition 1.1. A *crossed module* of groups is a map $\partial : N \to C$ together with an action $\varrho : C \to \mathrm{Aut}(N)$ of C on N such that
(i) $\partial(\varrho(c)(n)) = c\partial(n)c^{-1}$,
(ii) $\varrho(\partial(n))(m) = nmn^{-1}$
for $n, m \in N$, $c \in C$.

For a crossed module $(N, \partial, C, \varrho)$, we usually denote $\ker \partial$ by K, $\operatorname{im} \partial$ by J and $\operatorname{coker} \partial$ by G, and display the crossed module in the form

$$\begin{tikzcd} K \arrow[d, tail] & & \\ N \arrow[d, two heads, "\bar{\partial}"'] & & \\ J \arrow[r, tail] & C \arrow[r, two heads, "\chi"] & G \end{tikzcd} \qquad \varrho : C \to \mathrm{Aut}(N) \tag{3}$$

We describe (3) as a *crossed module on G* with kernel K. We will be considering crossed modules on a fixed G and with not only K but also N fixed, and so we define an *equivalence* $(N, \partial, C, \varrho) \to (N, \partial', C', \varrho')$ to be a morphism $\varphi : C \to C'$ such that $\varphi \circ \partial = \partial'$, $\chi' \circ \varphi = \chi$ and $\varrho' \circ \varphi = \varrho$. Weaker definitions than this are appropriate for other purposes (see, for example, Huebschmann [12]).

In this paper we are mainly concerned with the following class of examples:

Definition 1.2. Consider groups G and N. A *coupling* of G with N is a crossed module $(N, \partial, C, \varrho)$ on G whose kernel is ZN, the centre of N.

This usage is not standard; the word 'coupling' is taken from Robinson [21], who used it as an alternative to 'abstract kernel'. The usage here is partly for consistency with [17, Chapter IV].

We next demonstrate that couplings of G with N are in bijective correspondence with abstract kernels $G \to \mathrm{Out}(N)$. Some examples of crossed modules which are not couplings are given at the end of the section. For further examples, history and references see [3, §3].

Consider groups G and N and let $\gamma : G \to \mathrm{Out}(N)$ be an abstract kernel for G and N. Choose a lift $\bar{\gamma} : G \to \mathrm{Aut}(N)$ of γ with $\bar{\gamma}(1) = \mathrm{id}$; thus $\natural \circ \bar{\gamma} = \gamma$, where $\natural$ is the natural projection of $\mathrm{Aut}(N)$ onto $\mathrm{Out}(N)$. Define $\bar{R} : G \times G \to \mathrm{Inn}(N)$ by $\bar{R}(g_1, g_2) = \bar{\gamma}(g_1 g_2)^{-1} \bar{\gamma}(g_1) \bar{\gamma}(g_2)$, and define a multiplication on the set $C = G \times \mathrm{Inn}(N)$ by

$$(g_1, I_{n_1})(g_2, I_{n_2}) = (g_1 g_2, \bar{R}(g_1, g_2) I_{\bar{\gamma}(g_2)^{-1}(n_1)} I_{n_2}).$$

It is straightforward to verify that this makes C a group, and that $\mathrm{Inn}(N) \rightarrowtail C \twoheadrightarrow G$

is an extension, where the maps are the obvious ones. To finish, define an action ϱ of C on N by $\varrho(g, I_n) = \bar{\gamma}(g) \circ I_n$. Then (C, ϱ) is a coupling of G with N.

Conversely, let $(N, \partial, C, \varrho)$ be a coupling of G with N. Then $\varrho : C \to \mathrm{Aut}(N)$ maps $\mathrm{Inn}(N) \leq C$ to $\mathrm{Inn}(N) \leq \mathrm{Aut}(N)$ (by Definition 1.1(ii)) and so induces a morphism $\gamma : G \to \mathrm{Out}(N)$. It is straightforward to verify that any equivalent coupling yields the same γ. One now easily sees that there is a bijection between abstract kernels and equivalence classes of couplings. When G and N are understood, we will denote a coupling by (C, ϱ) and its equivalence class by $\langle C, \varrho \rangle$; we loosely refer to $\langle C, \varrho \rangle$ itself as a coupling.

Now if $N \rightarrowtail H \xrightarrow{\pi} G$ is an extension of G by N, the associated coupling consists of the left and bottom sides of the diagram

$$\begin{tikzcd} ZN \arrow[r, equal] \arrow[d, tail] & ZN \arrow[d, tail] & \\ N \arrow[r, tail] \arrow[d, "I"'] & H \arrow[r, two heads, "\pi"] \arrow[d, "\kappa"'] & G \arrow[d, equal] \\ \mathrm{Inn}\, N \arrow[r, tail] & C \arrow[r, two heads, "\chi"] & G \end{tikzcd} \tag{4}$$

together with the action ϱ of C on N defined by

$$\varrho(c)(n) = hnh^{-1} \quad \text{where } \kappa(h) = c.$$

It will be convenient later to regard C as the quotient of the semidirect product $H \ltimes \mathrm{Inn}(N)$ (where the action of H on $\mathrm{Inn}(N)$ is that induced from the inner automorphism action of H on its normal subgroup N) by the normal subgroup $\Delta = \{(n, I_n^{-1}) \mid n \in N\}$. We write elements of C as $\langle h, \varphi \rangle$, $h \in H$, $\varphi \in \mathrm{Inn}(N)$, and we now have

$$\langle h, \varphi \rangle = \langle hn, I_n^{-1} \circ \varphi \rangle \quad \text{for } n \in N$$

and

$$\langle h_1, \varphi_1 \rangle \langle h_2, \varphi_2 \rangle = \langle h_1 h_2, I_{h_2^{-1}} \varphi_1 I_{h_2} \varphi_2 \rangle.$$

The action of C on N is

$$\varrho(\langle h, \varphi \rangle)(n) = I_h(\varphi(n)).$$

Lastly, κ is $h \mapsto \langle h, \mathrm{id} \rangle$. One can easily check the consistency of this definition with that of the standard definition of the abstract kernel associated to an extension.

Definition 1.3. Let $\langle C, \varrho \rangle$ be a coupling of G with N. Then $N \rightarrowtail H \twoheadrightarrow G$ is a *lift of*, or an *operator extension for*, the coupling if there is a morphism $\kappa : H \to C$ such that

$$\begin{tikzcd}
N \arrow[r, tail] \arrow[d, "I"'] & H \arrow[r, two heads, "\pi"] \arrow[d, "\kappa"'] & G \arrow[d, equal] \\
\mathrm{Inn}(N) \arrow[r, tail] & C \arrow[r, two heads, "\chi"] & G
\end{tikzcd}$$

commutes, and such that $hnh^{-1} = \varrho(\kappa(h))(n)$ for $n \in N$, $h \in H$.

Note that this definition is independent of the particular representative (C, ϱ) chosen.

Definition 1.4. Let $\langle C, \varrho \rangle$ be a coupling of G with N. Two operator extensions

$$N \overset{\iota_1}{\rightarrowtail} H_1 \overset{\pi_1}{\twoheadrightarrow} G \quad \text{and} \quad N \overset{\iota_2}{\rightarrowtail} H_2 \overset{\pi_2}{\twoheadrightarrow} G$$

are *equivalent* if there is an isomorphism $\varphi : H_1 \to H_2$ such that $\pi_2 \circ \varphi = \pi_1$, $\varphi \circ \iota_1 = \iota_2$ and $\kappa_2 \circ \varphi = \kappa_1$.

The set of equivalence classes of operator extensions is denoted $\mathrm{Opext}(G, N, \langle C, \varrho \rangle)$.

Now consider a fixed coupling $\langle C, \varrho \rangle$ of G with N. Notice that G acts on ZN by $(g, z) \mapsto \varrho(c)(z)$, where $\chi(c) = g$; denote this action by ϱ^z. Choose a transversal $\sigma : G \to C$ for χ with $\sigma(1) = 1$, and define $R_\sigma : G \times G \to \mathrm{Inn}\, N$ by

$$R_\sigma(g_1, g_2) = \sigma(g_1 g_2)^{-1} \sigma(g_1) \sigma(g_2).$$

Choose a lift $\tilde{R} : G \times G \to N$ of R_σ with $\tilde{R}(g, 1) = \tilde{R}(1, g) = 1$ for all g; that $\tilde{R}$ is a lift of R_σ means that $I \circ \tilde{R} = R_\sigma$. Now define $e : G \times G \times G \to N$ by

$$e(g_1, g_2, g_3) = \tilde{R}(g_1 g_2, g_3) \varrho(\sigma(g_3)^{-1})(\tilde{R}(g_1, g_2)) \tilde{R}(g_2, g_3)^{-1} \tilde{R}(g_1, g_2 g_3)^{-1};$$

one sees that e takes values in ZN. It is routine to verify that e is a normalized 3-cocycle and that $[e] \in H^3(G, \varrho^z, ZN)$ is well-defined independently of all the choices involved in the definition of e.

If e is the zero cocycle, then a group structure is defined on the set $H = G \times N$ by

$$(g_1, n_1)(g_2, n_2) = (g_1 g_2, \tilde{R}(g_1, g_2) \varrho(\sigma(g_2)^{-1})(n_1) n_2),$$

and $N \rightarrowtail H \twoheadrightarrow G$ becomes an extension, where the maps are the obvious ones. Now define $\kappa : H \to C$ by $(g, n) \mapsto \sigma(g) I_n$; this map makes $N \rightarrowtail H \twoheadrightarrow G$ a lift of $\langle C, \varrho \rangle$. If e is merely cohomologous to zero, multiply the lift $\tilde{R}$ by any 2-cochain whose coboundary is e. This new lift defines a new obstruction cocycle e' which is precisely zero.

It is manifest that the coupling associated to an extension $N \rightarrowtail H \twoheadrightarrow G$ has $[e] = 0 \in H^3(G, ZN)$; one chooses σ to be $\kappa \circ \tilde{\sigma}$ for some transversal $\tilde{\sigma} : G \to H$, and one then chooses $\tilde{R}$ to be $R_{\tilde{\sigma}}$.

Now suppose that $\langle C, \varrho \rangle$ is a coupling with $[e] = 0 \in H^3(G, ZN)$. Define an action of $H^2(G, ZN)$ on $\mathrm{Opext}(G, N, \langle C, \varrho \rangle)$ as follows: Given an operator extension

$$N \rightarrowtail H \xrightarrow{\pi} G,$$

choose a transversal $\tilde{\sigma}: G \to H$ for π and let $\tilde{R}$ be the associated map

$$G \times G \to N, \qquad (g_1, g_2) \mapsto \tilde{\sigma}(g_1, g_2)^{-1} \tilde{\sigma}(g_1) \tilde{\sigma}(g_2).$$

Given $[f] \in H^2(G, ZN)$, observe that $\tilde{R}f$ is also a lift of $R = \kappa \circ \tilde{R}$ and defines the zero 3-cocycle; let

$$N \rightarrowtail H^f \xrightarrow{\pi^f} G$$

be the extension constructed from $\tilde{R}f$. It is routine to verify that this gives a well-defined action of $H^2(G, ZN)$ on $\mathrm{Opext}(G, N, \langle C, \varrho \rangle)$, and that the action is free and transitive. Furthermore, this action corresponds to the standard one.

One of the minor benefits of this treatment is that if $N = A$ is abelian, then a coupling of G with A is simply a representation of G on A, and the classification of operator extensions $A \rightarrowtail H \twoheadrightarrow G$ by $H^2(G, A)$ is thus easily obtainable from the nonabelian case.

All the foregoing applies to arbitrary crossed modules; a given crossed module

$$\begin{tikzcd}
K \arrow[d, tail] & & \\
N \arrow[d, "\bar{\partial}"'] & & & \varrho: C \to \mathrm{Aut}(N) \\
J \arrow[r, tail] & C \arrow[r, two heads, "\chi"] & G
\end{tikzcd}$$

defines an *obstruction class* $e \in H^3(G, \varrho^k, K)$; one may find a lifted extension $(N \rightarrowtail H \twoheadrightarrow G, \kappa)$ iff $e = 0$, and equivalence classes of lifted extensions are classified by $H^2(G, \varrho^k, K)$. The checking is straightforward.

This point of view was already taken by Dedecker [7]; in particular the view of an operator extension as a lift in (4) is a special case of a diagram in [7]; namely those extensions with 'crest' (crête) the identity. However, Dedecker was concerned with a much more general class of lifted extensions, to classify which he needed the full force of his nonabelian cohomology theory. One of the points of the classification given here is that one needs only the well-understood abelian cohomology.

As an introduction to Section 3, we very briefly consider crossed modules of Lie groups. All Lie groups are real, Hausdorff and have at most countably many components.

Definition 1.5. A *closed crossed module of Lie groups* is a crossed module $(N, \partial, C, \varrho)$ in which N and C are Lie groups, ϱ is a smooth representation of C on N, and ∂ is a smooth morphism with image closed in C.

The final condition ensures that the cokernel is Hausdorff. Notice that any non-closed subgroup of an abelian Lie group gives an example of a crossed module of Lie groups which is not a closed crossed module. By a *coupling* of Lie groups we will mean a coupling of the underlying groups which is a closed crossed module.

Couplings of Lie groups are easy to find. A natural example of a closed crossed module of Lie groups which is not generally a coupling is

$$
\begin{tikzcd}
\pi_1 G \arrow[d, tail] & & \\
\tilde{G}_0 \arrow[d, two heads] & & \mathrm{Ad}: G \to \mathrm{Aut}(\tilde{G}_0) \\
G_0 \arrow[r, tail] & G \arrow[r, two heads] & \pi_0 G
\end{tikzcd}
$$

where G is a Lie group, G_0 is its identity component, and $\tilde{G}_0$ is the universal covering of G_0. By Ad we here mean the adjoint action of G on its Lie algebra $\mathfrak{g}$, transferred to $\tilde{G}_0$. Although this is a Lie group problem, the obstruction class $e \in H^3(\pi_0 G, \pi_1 G)$ is the sole obstruction to the existence of a Lie group H with $H_0 \cong \tilde{G}_0$, $\pi_0 H \cong \pi_0 G$ and a suitable projection $H \twoheadrightarrow G$; see [22] and [5].

For general closed crossed modules of Lie groups, the definition of an obstruction class is a nontrivial matter and will be tackled elsewhere. Notice that the 'universal' example, $(N, I, \mathrm{Aut}(N), \mathrm{id})$, is not generally closed.

2. Lie groupoids and principal bundles

In this section we collect some necessary background on Lie groupoids and their relationship with principal bundles. We will be brief; for fuller accounts see, for example, [13, 17, 18]. Throughout the rest of this paper all manifolds are real, para-compact, C^∞, and of constant dimension.

Definition 2.1. A *groupoid* on base B is a set Ω together with maps $\alpha: \Omega \to B$, $\beta: \Omega \to B$ and $\varepsilon: B \to \Omega$, $x \mapsto \tilde{x}$, called respectively the *source*, the *target* and the *object inclusion map*, and a partial multiplication $\kappa: \Omega * \Omega \to \Omega$, $(\eta, \xi) \mapsto \eta\xi$, where $\Omega * \Omega = \{(\eta, \xi) \in \Omega \times \Omega \mid \alpha\eta = \beta\xi\}$, such that

(i) $\alpha(\eta\xi) = \alpha(\xi)$ and $\beta(\eta\xi) = \beta(\eta)$ for all $(\eta, \xi) \in \Omega * \Omega$;

(ii) $\alpha(\tilde{x})=\beta(\tilde{x})=x$ for all $x\in B$;
(iii) $\zeta(\eta\xi)=(\zeta\eta)\xi$ whenever $\alpha\eta=\beta\xi$ and $\alpha\zeta=\beta\eta$;
(iv) $\xi\tilde{x}=\xi$ and $\tilde{y}\xi=\xi$, where $x=\alpha\xi$ and $y=\beta\xi$, for all $\xi\in\Omega$;
(v) For each $\xi\in\Omega$ there exists a unique $\xi^{-1}\in\Omega$ such that $\alpha(\xi^{-1})=\beta(\xi)$, $\beta(\xi^{-1})=\alpha(\xi)$ and $\xi\xi^{-1}=\widetilde{\beta(\xi)}$, $\xi^{-1}\xi=\widetilde{\alpha(\xi)}$.

The set B can be identified with $\{\tilde{x}\,|\,x\in B\}\subseteq\Omega$ and may be thought of as the set of identity elements for the multiplication in Ω.

For differential geometric purposes, a prototypical example of a groupoid is given by a manifold B and the set Ω of linear isomorphisms between the various tangent spaces to B. Then for $\xi: T(B)_x\to T(B)_y$ in Ω, define $\alpha(\xi)=x$, and $\beta(\xi)=y$. For $x\in B$, let $\tilde{x}$ be

$$\mathrm{id}_{T(B)_x}.$$

Lastly, let κ be the standard composition of maps.

Example 2.2. Let $P(B,G,p)$ be a principal bundle, and consider the diagonal action of G on $P\times P$, namely $(v,u)g=(vg,ug)$. Denote the orbit of (v,u) by $\langle v,u\rangle$ and the set of orbits by

$$\frac{P\times P}{G}.$$

It is easy to see that this set is a smooth manifold; indeed

$$P\times P\left(\frac{P\times P}{G},G\right)$$

is itself a principal bundle. Now

$$\frac{P\times P}{G}$$

is a groupoid on B under the structure

$$\alpha(\langle v,u\rangle)=p(u),\qquad \beta(\langle v,u\rangle)=p(v);$$

$$\tilde{x}=\langle u,u\rangle\quad\text{for any } u\in p^{-1}(x)$$

and

$$\langle w,v\rangle\langle v,u\rangle=\langle w,u\rangle.$$

To appreciate the definition of this multiplication, take any two elements $\eta=\langle v_2,u_2\rangle$ and $\xi=\langle v_1,u_1\rangle$ in

$$\frac{P\times P}{G}$$

with $\alpha(\eta)=\beta(\xi)$. Then $p(u_2)=p(v_1)$ so there exists a unique $g\in G$ with $u_2=v_1g$.

Now $\langle v_2, u_2\rangle = \langle v_2 g^{-1}, v_1\rangle$ and so the definition of the multiplication applies and we get

$$\langle v_2, u_2\rangle\langle v_1, u_1\rangle = \langle v_2 g^{-1}, u_1\rangle.$$

Example 2.3. Consider a set B and a group G. The set $B\times G\times B$ carries the groupoid structure

$$\alpha(y,g,x)=x, \qquad \beta(y,g,x)=y; \qquad \tilde{x}=(x,1,x)$$

and

$$(z,h,y)(y,g,x)=(z,hg,x).$$

Applying Example 2.2 to the trivial bundle $B\times G(B,G)$, for a manifold B and Lie group G, one obtains precisely this groupoid, and it is accordingly called the *trivial groupoid* on B with group G.

Taking $G=1$, we refer to $B\times B$ as the *pair groupoid* on B.

Likewise, applying Example 2.2 to the universal covering $\tilde{B}(B,\pi_1 B)$ of a connected manifold B yields the fundamental groupoid of B.

All of these examples are transitive in the following sense:

Definition 2.4. A groupoid Ω on B is *transitive* if for all $x,y\in B$ there is an element $\xi\in\Omega$ such that $\alpha\xi=x$, $\beta\xi=y$.

Transitivity ensures considerable homogeneity in a groupoid as we shall see.

Definition 2.5. Consider groupoids Ω on B and Ω' on B'. A *morphism* from Ω to Ω' is a pair of maps $\varphi:\Omega\to\Omega'$, $\varphi_0:B\to B'$ such that $\alpha'\circ\varphi=\varphi_0\circ\alpha$, $\beta'\circ\varphi=\varphi_0\circ\beta$, and $\varphi(\eta\xi)=\varphi(\eta)\varphi(\xi)$ whenever $(\eta,\xi)\in\Omega*\Omega$.

In the case where $B=B'$ and $\varphi_0=\mathrm{id}$, we say that φ is a morphism *over* B, or is *base-preserving*.

It follows that $\varphi\circ\varepsilon=\varepsilon'\circ\varphi_0$. As a single example, it is easy to see that a morphism of principal bundles $\varphi(\varphi_0,f):P(B,G)\to P'(B',G')$ induces a morphism of the associated groupoids

$$\frac{P\times P}{G}\to\frac{P'\times P'}{G'}, \qquad \langle v,u\rangle\mapsto\langle\varphi(v),\varphi(u)\rangle.$$

We are, of course, primarily interested in groupoids with an additional smooth structure.

Definition 2.6. A *differentiable groupoid* is a groupoid Ω on base B together with smooth structures on Ω and B such that $\alpha,\beta:\Omega\to B$ are surjective submersions and $\varepsilon:B\to\Omega$ and $\kappa:\Omega*\Omega\to\Omega$ are smooth.

A *morphism* of differentiable groupoids $\varphi:\Omega\to\Omega'$, $\varphi_0:B\to B'$ is a morphism of the underlying groupoids such that φ (and hence φ_0) are smooth.

For the smoothness of κ, notice that the conditions on α and β ensure that $\Omega * \Omega$ is a closed, embedded submanifold of $\Omega \times \Omega$. It also follows, by an argument similar to that used for Lie groups, that $\xi \mapsto \xi^{-1}$, $\Omega \to \Omega$ is smooth, and hence a diffeomorphism.

The concept of differentiable groupoid was introduced by Ehresmann in the 1950's; see also Pradines [19a]. Here we are concerned only with those differentiable groupoids which satisfy a local triviality condition (Definition 2.7 below) and to express this we need some further preliminaries.

For any groupoid Ω and any $x, y \in B$, write $\Omega_x = \alpha^{-1}(x)$, $\Omega^y = \beta^{-1}(y)$ and $\Omega_x^y = \Omega_x \cap \Omega^y$. Call Ω_x the *α-fibre* over x, and Ω^y the *β-fibre* over y. For each $x \in B$, the multiplication in Ω gives a group structure on Ω_x^x; call it the *vertex group* at x. This group acts freely to the right on Ω_x, and the orbits of the action are equal to the fibres of β_x, the restriction to Ω_x of β.

Each element $\xi \in \Omega_x^y$ defines a *right-translation* $R_\xi : \Omega_y \to \Omega_x$, $\eta \mapsto \eta\xi$, and a so-called *inner automorphism* $I_\xi : \Omega_x^x \to \Omega_y^y$, $\lambda \mapsto \xi\lambda\xi^{-1}$, which is an isomorphism of the vertex groups. Thus in a transitive groupoid the vertex groups are all isomorphic and, further, the

$$R_{\xi^{-1}}(\mathrm{id}_B, I_\xi) : \Omega_x(B, \Omega_x^x, \beta_x) \to \Omega_y(B, \Omega_y^y, \beta_y)$$

are isomorphisms of the 'set-theoretic principal bundles' at x and y.

Definition 2.7. A *Lie groupoid* is a transitive differentiable groupoid Ω on a base manifold B such that $\beta_x : \Omega_x \to B$ is a submersion (that is, is of maximal rank) for some, and hence every, $x \in B$.

It follows that each $\Omega_x(B, \Omega_x^x, \beta_x)$ is a principal bundle in the standard sense; further, all these principal bundles are isomorphic under the $R_\xi^{-1}(\mathrm{id}_B, I_\xi)$.

The local triviality properties of a Lie groupoid Ω are thus expressed by local sections of any fixed vertex bundle. Namely, we choose $b \in B$ as a reference-point and consider maps

$$\sigma_i : U_i \to \Omega_b \quad \text{with} \quad \beta_b \circ \sigma_i = \mathrm{id}_{U_i},$$

where $\{U_i\}$ is an open cover of B. We refer to such a family $\{\sigma_i : U_i \to \Omega_b\}$ as a *section-atlas* for Ω, or for $\Omega_b(B, \Omega_b^b)$. The associated *transition functions* can now be expressed simply as $s_{ij} : U_{ij} \to \Omega_b^b$, $x \mapsto \sigma_i(x)^{-1}\sigma_j(x)$ where $U_{ij} = U_i \cap U_j \neq \emptyset$; these are the transition functions for $\Omega_b(B, \Omega_b^b)$ in the standard sense. Each section $\sigma_i : U_i \to \Omega_b$ induces an isomorphism of Lie groupoids over U_i from the trivial groupoid $U_i \times \Omega_b^b \times U_i$ to the restriction

$$\Omega_{U_i}^{U_i} = \alpha^{-1}(U_i) \cap \beta^{-1}(U_i)$$

by $(y, g, x) \mapsto \sigma_i(y) g \sigma_i(x)^{-1}$. Further details of this formalism for Lie groupoids are given in [17, II§2].

A remarkable theorem of Pradines (see [17, III§1] or [20]) asserts that for an arbitrary differentiable groupoid, all the maps $\beta_x : \Omega_x \to B$ are of locally constant rank. From this it follows that, for the standard concept of manifold with which we are dealing here, a transitive differentiable groupoid is automatically Lie. However, for more general forms of the manifold concept this is not the case.

This usage of the term 'Lie groupoid' seems to be due to Ngô Van Que [18], following Matushima.

We have seen that to a Lie groupoid Ω on base B there is associated a collection of principal bundles $\Omega_x(B, \Omega_x^x)$, all of which are mutually isomorphic, but not — in general — in any canonical way. Conversely, Example 2.2 associates to any principal bundle $P(B, G, p)$ a groupoid

$$E(P) = \frac{P \times P}{G},$$

sometimes called the *Ehresmann groupoid* of $P(B, G)$, which is easily seen to be Lie. In fact if one chooses a reference point $u_0 \in P$, then the vertex principal bundle

$$E(P)_{x_0}(B, E(P)_{x_0}^{x_0})$$

over $x_0 = p(u_0)$ is isomorphic to $P(B, G)$ under the maps

$$P \to E(P)_{x_0}, \qquad u \mapsto \langle u, u_0 \rangle$$

and

$$G \to E(P)_{x_0}^{x_0}, \qquad g \mapsto \langle u_0 g, u_0 \rangle.$$

Conversely, any Lie groupoid Ω is isomorphic to the Lie groupoid

$$E(\Omega_x) = \frac{\Omega_x \times \Omega_x}{\Omega_x^x}$$

associated to any of its vertex bundles $\Omega_x(B, \Omega_x^x)$, under

$$E(\Omega_x) \to \Omega, \qquad \langle \eta, \xi \rangle \mapsto \eta \xi^{-1}.$$

It is a fact of life that these two constructions do not give a complete bijective correspondence between the concepts of Lie groupoid and principal bundle, but depend upon the choice of reference-points. Many large classes of principal bundles — for example, all frame bundles associated to structured manifolds — themselves depend on a rather arbitrary choice of reference point (the choice of a specific fibre as typical fibre is effectively the choice of a reference point) and in these cases the groupoid seems the more natural object. On the other hand the principal bundle $G(G/H, H)$ defined by a Lie group G and a closed subgroup H is more natural than the corresponding groupoid.

Changing the reference-point u_0 in a principal bundle $P(B, G)$ to a point $u_0 g$ in the same fibre leads to automorphisms of $P(B, G)$ of the form $R_{g^{-1}}(\mathrm{id}, I_g)$. For details see, for example, [17, II§1].

Now a principal bundle comes with its structure group readily displayed, but for a general Lie groupoid Ω there is no natural way to single out a particular vertex group. We therefore consider the union of all the vertex groups,

$$\bigcup_{x\in B} \Omega_x^x,$$

which we denote by $I\Omega$ and call the *gauge group bundle* of Ω (called in [17] the inner group bundle of Ω). It is a Lie group bundle in the sense of the following definition, which goes back at least to [8]:

Definition 2.8. A *Lie group bundle*, or *LGB*, is a triple (M, p, B) in which $p: M \to B$ is a smooth surjective submersion, each $M_x = p^{-1}(x)$, $x \in B$, has a Lie group structure, and there is a Lie group G, called the *fibre-type* of M, with respect to which p is locally trivial in the sense that there is an open cover $\{U_i\}$ of B and charts

$$\psi_i : U_i \times G \to M_{U_i}$$

for which each $\psi_{i,x}: G \to M_x$, $g \to \psi_i(x, g)$, is a Lie group isomorphism.

A *morphism* of LGB's from (M, p, B) to (M', p', B') is a pair of maps $\varphi: M \to M'$, $\varphi_0: B \to B'$ such that $p' \circ \varphi = \varphi_0 \circ p$ and each

$$\varphi_x : M_x \to M'_{\varphi_0(x)}, \quad x \in B,$$

is a Lie group morphism.

It is worth noting that an LGB is itself a differentiable groupoid on B.

For a Lie groupoid Ω, fix $b \in B$, let $\{\sigma_i : U_i \to \Omega_b\}$ be a system of local sections for the vertex principal bundle at b, and consider the associated inner automorphisms

$$I_{\sigma_i} : U_i \times \Omega_b^b \to I\Omega\,|_{U_i}, \qquad (x, g) \mapsto \sigma_i(x) g \sigma_i(x)^{-1}.$$

These give an LGB atlas for $I\Omega$.

If one starts with a principal bundle $P(B, G)$, the inner group bundle of

$$\frac{P \times P}{G}$$

is naturally isomorphic to

$$\frac{P \times G}{G}$$

the fibre bundle associated to $P(B, G)$ through the inner automorphism action of G on itself. Thus elements of

$$\frac{P \times G}{G}$$

are orbits $\langle u, g\rangle$ of $P \times G$ under the action $(u, g)h = (uh, h^{-1}gh)$. The group structure in a fibre

$$\left.\frac{P\times G}{G}\right|_x$$

is then the natural one, namely $\langle u, g_1\rangle\langle u, g_2\rangle = \langle u, g_1g_2\rangle$. This is the bundle whose space of sections forms what is called, in the physics literature, the gauge group of $P(B, G)$.

The structure group of a principal bundle is sometimes thought of as the 'kernel' of the bundle projection. In a similar way, but much more precisely, the gauge group bundle of a Lie groupoid is the kernel of a certain natural morphism. Namely, for a differentiable groupoid Ω on B, define $(\beta, \alpha): \Omega \to B \times B$, $\xi \mapsto (\beta\xi, \alpha\xi)$. This is a morphism of differentiable groupoids over B into the pair groupoid $B \times B$. In [17] we have called it the *anchor* of Ω. The following result is easy to check, using local sections of the maps concerned:

Proposition 2.9. *A differentiable groupoid Ω on B is Lie iff the anchor $(\beta, \alpha): \Omega \to B \times B$ is a surjective submersion.* $\square$

Thus for a Lie groupoid Ω we have a sequence

$$I\Omega \rightarrowtail \Omega \xrightarrow{(\beta,\alpha)} B \times B \tag{5}$$

which is exact not only in the obvious algebraic sense but also in the smooth sense that $I\Omega \to \Omega$ is a closed embedding and (β, α) is of maximal rank.

Our purpose in Section 3 is to show that (5) and its principal bundle version,

$$\frac{P\times G}{G} \rightarrowtail \frac{P\times P}{G} \twoheadrightarrow B \times B \tag{6}$$

are classified by a suitable abelian Čech cohomology.

3. The classification

We first need some basic algebraic constructions. For the general algebra of groupoids see Higgins [11]; the constructions we need here are summarized in [17, Chapter I].

For two transitive groupoids, Ω, Ω' on the same base B, and a base-preserving morphism $\varphi: \Omega \to \Omega'$, it is easy to see that the *kernel* of φ, by which we mean $\{\xi \in \Omega \mid \exists x \in B: \varphi(\xi) = \tilde{x}\}$, is entirely contained in $I\Omega$. Therefore, in the definition of a quotient groupoid, it will suffice to consider normal subgroupoids of Ω which are contained in $I\Omega$.

Definition 3.1. Let Ω be a transitive groupoid on B.

(i) A *normal subgroupoid* of Ω is a subset M of $I\Omega$ which contains $\{\tilde{x} \mid x \in B\}$, which is closed under the multiplication and inversion in Ω, and which is such that $\xi m \xi^{-1} \in M$ for every $m \in M$ and every $\xi \in \Omega$ for which $\xi m \xi^{-1}$ is defined.

(ii) Let M be a normal subgroupoid of Ω. Then the *quotient groupoid* Ω/M is the groupoid on B whose elements are the cosets $\langle \xi \rangle = \{\xi m \mid m \in M, \alpha\xi = \beta m\}$ and whose structure is given by $\bar{\alpha}(\langle \xi \rangle) = \alpha(\xi)$, $\bar{\beta}(\langle \xi \rangle) = \beta(\xi)$, $\bar{\varepsilon}(x) = \langle \tilde{x} \rangle$, and $\langle \eta \rangle \langle \xi \rangle = \langle \eta \xi \rangle$ whenever defined.

It is straightforward to check that Ω/M is indeed a groupoid and that the natural map $\natural : \Omega \to \Omega/M$, $\xi \mapsto \langle \xi \rangle$, is a morphism over B with kernel M.

Now suppose that Ω is a Lie groupoid and that M is a normal subgroupoid of Ω which is a closed embedded submanifold of Ω. Then, because the inner automorphism of Ω restrict to M, it is automatic that M is an LGB. One can prove easily enough that the graph of the equivalence relation "$\xi \simeq \eta \Leftrightarrow \exists m \in M : \xi = \eta m$" is a closed embedded submanifold of $\Omega \times \Omega$ and it therefore follows, by the criterion of Godement, that the quotient manifold Ω/M exists (see [17, III 1.32], for example). Now by working locally one can see that this structure makes Ω/M a Lie groupoid (compare [17, II 2.15]).

We denote this situation by the exact sequence $M \rightarrowtail \Omega \twoheadrightarrow \Omega/M$.

Definition 3.2. Let Ω be a Lie groupoid on a base manifold B, and let (M, p, B) be an LGB on B. A *representation* of Ω on M is a smooth map $\varrho : \Omega * M \to M$, where $\Omega * M$ is the pullback manifold $\{(\xi, m) \in \Omega \times M \mid \alpha\xi = pm\}$, such that

(i) $p(\varrho(\xi, m)) = \beta(\xi)$ for $(\xi, m) \in \Omega * M$;

(ii) $\varrho(\eta, \varrho(\xi, m)) = \varrho(\eta\xi, m)$ for all m, η, ξ such that $(\xi, m) \in \Omega * M$ and $(\eta, \xi) \in \Omega * \Omega$;

(iii) $\varrho(\widetilde{pm}, m) = m$ for all $m \in M$;

(iv) $\varrho(\xi) : M_{\alpha\xi} \to M_{\beta\xi}$, $m \mapsto \varrho(\xi, m)$, is a Lie group isomorphism for all $\xi \in \Omega$.

The concept of a groupoid representation on a fibered manifold is due to Ehresmann. To an LGB M one can associate a Lie groupoid $\Pi(M)$ on the same base B, whose elements are the Lie group isomorphisms between the fibres of M, and a representation of Ω on M can then be considered to be a morphism of Lie groupoids $\Omega \to \Pi(M)$. See, for example, [17]. We do not need this formulation here.

Definition 3.3. A *closed crossed module* of Lie groupoids is a quadruple $(M, \partial, \Xi, \varrho)$, where Ξ is a Lie groupoid on base B, where M is an LGB on the same base, where $\partial : M \to \Xi$ is a morphism of differentiable groupoids over B, and where ϱ is a representation of Ξ on M, all such that

(i) $\partial(\varrho(\xi, m)) = \xi \partial(m) \xi^{-1}$ for all $(\xi, m) \in \Xi * M$;

(ii) $\varrho(\partial(m), m') = m m' m^{-1}$ for all $m, m' \in M$ with $p(m) = p(m')$; and

(iii) $\mathrm{im}(\partial)$ is a closed embedded submanifold of Ξ.

Crossed modules of (set-theoretic) groupoids were considered by Brown and Higgins [4]; their Theorem 6.2 shows (as a special case) that crossed modules of groupoids are equivalent to double groupoids of a certain type.

Notice that $\mathrm{im}(\partial)$ must lie entirely in $I\Xi$, and is normal in Ξ. The image of a morphism of LGB's may easily fail to be an LGB itself, since an arbitrary morphism may (for example) be constant on one fibre and injective on all others. That this cannot happen here is assured by the normalcy of $\mathrm{im}(\partial)$. We denote $\mathrm{im}(\partial)$ by J and the quotient Ξ/J by Ω. Regarding ∂ temporarily as a morphism of LGB's $\bar{\partial}: M \to J$ over B, and (as such) a surjective submersion, it has a kernel LGB on B, which we denote by K. We express all this in the diagram

$$\begin{tikzcd} K \arrow[d, tail] & & \\ M \arrow[d, two heads, "\bar{\partial}"'] & & & \varrho: \Xi * M \to M \\ J \arrow[r, tail] & \Xi \arrow[r, two heads, "\chi"] & \Omega \end{tikzcd}$$

As in Section 1, we say that $(M, \partial, \Xi, \varrho)$ is a closed crossed module *on* Ω with kernel K. Notice that ϱ induces a representation of Ω on K, which we denote by ϱ^K.

The definition of *equivalence* of closed crossed modules of Lie groupoids is identical in form to that of Section 1, and we denote an equivalence class of closed crossed modules by $\langle M, \partial, \Xi, \varrho \rangle$. Similarly, by a *coupling* of a Lie groupoid Ω with an LGB M on the same base, we mean a closed crossed module $\langle M, I, \Xi, \varrho \rangle$ on Ω with kernel ZM, the centre of M. Here ZM is the kernel of $I: M \to \mathrm{Aut}(M)$, $m \mapsto (m' \mapsto mm'm^{-1})$, which is easily seen to be a base-preserving LGB morphism with well-defined kernel and image. $\mathrm{Aut}(M)$ itself is the LGB whose fibre over $x \in B$ is the Lie group $\mathrm{Aut}(M_x)$ and whose charts

$$\psi: U_i \times \mathrm{Aut}(G) \to \mathrm{Aut}(M)_{U_i}$$

are constructed from charts

$$\psi_i: U_i \times G \to M_{U_i}$$

for M by $\psi_{i,x}(\varphi) = \psi_{i,x} \circ \varphi \circ \psi_{i,x}^{-1}$. When Ω and M are understood, we denote a coupling by $\langle \Xi, \varrho \rangle$ as in Section 1. Lastly, given an extension

$$M \rightarrowtail \Phi \xrightarrow{\pi} \Omega$$

of a Lie groupoid Ω by an LGB M on the same base, the associated coupling is $(M, I, \Phi/ZM, \varrho)$ obtained from the diagram

$$\begin{tikzcd}
ZM \arrow[r, equal] \arrow[d, tail] & ZM \arrow[d, tail] & \\
M \arrow[r, tail] \arrow[d, "I"', two heads] & \Phi \arrow[r, two heads, "\pi"] \arrow[d, "\kappa", two heads] & \Omega \arrow[d, equal] \\
\mathrm{Inn}(M) \arrow[r, tail] & \Phi/ZM \arrow[r, two heads, "\chi"] & \Omega
\end{tikzcd}$$

exactly as in Section 1.

We propose here to study the lifting problem only for those closed crossed modules with $\Omega = B \times B$, the pair Lie groupoid on B. That this is already an interesting question is, we believe, shown by what follows. The general case is a problem of a wholly different order, and will be tackled elsewhere. As in Section 1, we give the calculations only in the case of couplings.

Suppose, therefore, that we have a manifold B, an LGB (M, p, B) on B, and a coupling $\langle \Xi, \varrho \rangle$ of $B \times B$ with M. Thus

$$\mathrm{Inn}(M) \rightarrowtail \Xi \xrightarrow{\chi} B \times B$$

displays Ξ as a Lie groupoid on B with inner group bundle isomorphic to $\mathrm{Inn}(M)$.

Fix a reference-point $b \in B$, and write $N = M_b$. Choose a simple open cover $\mathscr{U} = \{U_i\}$ of B, and a section-atlas $\{\sigma_i : U_i \to \Xi_b\}$ for Ξ. Let $s_{ij} : U_{ij} \to \mathrm{Inn}(N)$ be the associated transition functions. Since U_{ij} is contractible, there exist maps $\hat{s}_{ij} : U_{ij} \to N$ such that $I \circ \hat{s}_{ij} = s_{ij}$. Define $e_{ijk} : U_{ijk} \to N$ by

$$e_{ijk}(x) = \hat{s}_{jk}(x)\hat{s}_{ik}(x)^{-1}\hat{s}_{ij}(x);$$

evidently $I \circ e_{ijk}$ is constant at id_N, and so e_{ijk} takes values in ZN. A routine calculation shows that, for $U_{ijkl} \neq \emptyset$,

$$e_{jkl} - e_{ikl} + e_{ijl} - e_{ijk} \equiv 0 \in ZN$$

and so e is a 2-cocycle in $\check{H}^2(\mathscr{U}, ZN)$, the Čech cohomology of B with respect to the cover $\mathscr{U}$, and with coefficients in the sheaf of germs of local maps from B to ZN.

It is trivial to see that if a second family of lifts $U_{ij} \to N$ of the s_{ij} is chosen, then the resulting cocycle is cohomologous to e. More generally, if $\{\sigma_i' : U_i \to \Xi_b\}$ is a second section-atlas for Ξ with respect to the same simple open cover $\mathscr{U}$, then $\sigma_i' = \sigma_i r_i$ for maps $r_i : U_i \to \mathrm{Inn}(N)$, and since the U_i are contractible, one can write $r_i = I \circ n_i$ for maps $n_i : U_i \to N$. Given lifts $\hat{s}_{ij}$ of the original transition functions $s_{ij} : U_{ij} \to \mathrm{Inn}(N)$, it now follows that $\hat{s}_{ij}' = n_i^{-1}\hat{s}_{ij} n_j$ are lifts of the transition functions s_{ij}' for $\{\sigma_i'\}$. The cocycle defined by these $\hat{s}_{ij}'$ is now $e_{ijk}' = n_j^{-1} e_{ijk} n_j$, and since e takes values in ZN, the two cocycles are equal. There is therefore a well-defined element $e \in \check{H}^2(\mathscr{U}, ZN)$ and, by the usual inductive process, of $\check{H}^2(B, ZN)$. We call $e \in \check{H}^2(B, ZN)$ the *obstruction class* of the coupling $\langle \Xi, \varrho \rangle$.

Theorem 3.4. *Continuing the above notation, there exists a Lie groupoid Ω on B with gauge group bundle isomorphic to M and coupling $\langle \Xi, \varrho \rangle$ iff $e = 0 \in \check{H}^2(B, ZN)$.*

Proof. Assume that $e = 0$. Take a simple open cover $\mathscr{U} = \{U_i\}$ of B and a section-atlas $\{\sigma_i : U_i \to \Xi_b\}$ for Ξ. Take a family of lifts $\tilde{s}_{ij} : U_{ij} \to N$ of the transition functions s_{ij}. Then there exists a cochain $\{z_{ij}\}$ in $\check{Z}^1(\mathscr{U}, ZN)$ with $\tilde{s}_{jk}\tilde{s}_{ik}^{-1}\tilde{s}_{ij} = z_{jk} - z_{ik} + z_{ij}$ and so the maps $\hat{s}_{ij} = \tilde{s}_{ij} z_{ij}^{-1} : U_{ij} \to N$ form an N-valued cocycle on B. Note that $I \circ \hat{s}_{ij} = I \circ \tilde{s}_{ij} = s_{ij}$.

In [17, II 2.19] we demonstrated the construction of a Lie groupoid directly from a cocycle. Applying this to $\{\hat{s}_{ij}\}$, take X to be the disjoint union

$$\coprod_{i,j} (U_j \times N \times U_i)$$

and define on X the equivalence relation

$$(j, y, n, x, i) \sim (j', y', n', x', i') \quad \Leftrightarrow \quad y = y', \quad x = x' \quad \text{and}$$
$$n' = \hat{s}_{j'j}(y) n \hat{s}_{ii'}(x).$$

Let Ω be $X/\sim$, and denote elements of Ω by $\langle j, y, n, x, i \rangle$. The groupoid structure on Ω is

$$\alpha(\langle j, y, n, x, i \rangle) = x, \qquad \beta(j, y, n, x, i \rangle) = y,$$

$$\tilde{x} = \langle i, x, 1, x, i \rangle \quad \text{for any } i \text{ with } x \in U_i,$$

and multiplication

$$\langle k, z, n_2, y, j_2 \rangle \langle j_1, y, n_1, x, i \rangle = \langle k, z, n_2 s_{j_2 j_1}(y) n_1, x, i \rangle.$$

It is routine to verify that Ω is a transitive groupoid on B. Place a manifold structure on Ω using the charts

$$U_j \times N \times U_i \to \Omega_{U_i}^{U_j}, \qquad (y, n, x) \mapsto \langle j, y, n, x, i \rangle.$$

Then it is straightforward to verify that Ω is a Lie groupoid on the manifold B.

Define $\iota : M \to \Omega$ by mapping $\varrho(\sigma_i(x), n) \in M_x$ to $\langle i, x, n, x, i \rangle$. Notice that the representation ϱ in fact induces an atlas of LGB charts $\varrho(\sigma_i(x))$ for M. Thus every element of M, say $m \in M_x$, can be represented as $\varrho(\sigma_i(x), n)$ for any i with $x \in U_i$. It is trivial to check that ι is well defined, and an isomorphism of LGB's over B onto $I\Omega$. Thus we have

$$M \xrightarrow{\iota} \Omega \xrightarrow{(\beta, \alpha)} B \times B.$$

Define $\kappa : \Omega \to \Xi$ by $\langle j, y, n, x, i \rangle \mapsto \sigma_j(y) I_n \sigma_i(x)^{-1}$. Again one checks easily that κ is well defined, a surjective submersion and a morphism of Lie groupoids over B.

To see that

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "I"'] & \Omega \arrow[d, "\kappa"] \\ \mathrm{Inn}(M) \arrow[r] & \Xi \end{tikzcd}$$

commutes, recall that $I_{\varrho(\xi)(m)}=\xi I_m \xi^{-1}$ for $\xi\in\Xi$, $m\in M_{\alpha\xi}$; this is Definition 3.3(i). Taking $\xi=\sigma_i(x)$ and $m=n\in N=M_b$, this gives

$$I_{\varrho(\sigma_i(x),n)}=(\kappa\circ\iota)(\varrho(\sigma_i(x),n)),$$

as required.

It remains to verify that the action of Ξ on M induced by the diagram (compare (4) and Definition 1.3)

$$\begin{tikzcd} ZM \arrow[r, equal] \arrow[d, tail] & ZM \arrow[d, tail] & \\ M \arrow[r, tail, "\iota"] \arrow[d, "I"'] & \Omega \arrow[r, two heads, "(\beta,\alpha)"] \arrow[d, "\kappa"] & B\times B \arrow[d, equal] \\ \mathrm{Inn}(M) \arrow[r, tail] & \Xi \arrow[r, two heads, "\chi"] & B\times B \end{tikzcd}$$

coincides with the given ϱ. Take $\omega\in\Omega$, say $\omega=\langle j,y,n,x,i\rangle$, and $m\in M_{\alpha\omega}$, say $m=\varrho(\sigma_{i'}(x),n')$; it is no loss of generality to assume that $i'=i$. Now $\omega\iota(m)\omega^{-1}=\langle j,y,nn'n^{-1},y,j\rangle$, by the definition of ι and the multiplication in Ω. On the other hand, $\kappa(\omega)$ is equal to $\sigma_j(y)I_n\sigma_i(x)^{-1}$ and so $\varrho(\kappa(\omega),m)=\varrho(\sigma_j(y)I_n,n')=\varrho(\sigma_j(y),nn'n^{-1})$, using Definition 3.3(ii). So $\omega\iota(m)\omega^{-1}=\iota(\varrho(\kappa(\omega),m))$, as required.

This completes the proof that

$$\left(M \overset{\iota}{\rightarrowtail} \Omega \xrightarrow{(\beta,\alpha)} B\times B,\kappa\right)$$

is a lift of the coupling $\langle\Xi,\varrho\rangle$.

The converse is a trivial verification. □

For the next result, consider a coupling of $B\times B$ with M for which $e\in\check{H}^2(B,ZN)$ is zero. As in Section 1, let $\mathrm{Opext}(B\times B,M,\langle\Xi,\varrho\rangle)$ denote the set of equivalence classes of operator Lie groupoids

$$\left(M\rightarrowtail\Omega\xrightarrow{(\beta,\alpha)} B\times B,\kappa\right)$$

for the coupling $\langle \Xi, \varrho \rangle$ (compare Definitions 1.3 and 1.4). We define an action of $\check{H}^1(B, ZN)$ on $\mathrm{Opext}(B \times B, M, \langle \Xi, \varrho \rangle)$.

Consider an operator Lie groupoid

$$\left(M \stackrel{\iota}{\rightarrowtail} \Omega \xrightarrow{(\beta,\alpha)} B \times B, \kappa\right)$$

for $\langle \Xi, \varrho \rangle$, and an element $f \in \check{H}^1(B, ZN)$. Let $\mathscr{U} = \{U_i\}$ be a simple open cover for B. Now principal bundles on a contractible base are trivializable and Lie groupoids, by the remarks in Section 2, are also, so there exists a section-atlas $\{\hat{\sigma}_i : U_i \to \Omega_b\}$ for Ω. Here b is a reference-point in B and $N = M_b$. Let $\{\hat{s}_{ij} : U_{ij} \to N\}$ denote the transition functions for $\{\hat{\sigma}_i\}$. Let $\{f_{ij} : U_{ij} \to ZN\}$ be a cocycle representing f with respect to $\mathscr{U}$. Then the maps $\hat{s}_{ij} f_{ij} : U_{ij} \to N$ satisfy the cocycle equation and $I \circ (\hat{s}_{ij} f_{ij}) = I \circ \hat{s}_{ij}$. By following through the proof of Theorem 3.4, one can see that Ω^f, the Lie groupoid constructed from the $\hat{s}_{ij} f_{ij}$, is an operator Lie groupoid for $\langle \Xi, \varrho \rangle$.

We prove that this action $\Omega \mapsto \Omega^f$ is well defined. Suppose that

$$\left(M \stackrel{\iota'}{\rightarrowtail} \Omega' \xrightarrow{(\beta',\alpha')} B \times B, \kappa'\right)$$

is a second operator Lie groupoid, equivalent to the first under $\varphi : \Omega \to \Omega'$. Choose a section-atlas $\{\hat{\sigma}_i' : U_i \to \Omega_b'\}$ for Ω' with respect to the same $\mathscr{U}$ and b, and write $\hat{\sigma}_i' = (\varphi \circ \hat{\sigma}_i) r_i$, where $r_i : U_i \to N$. (Here we are omitting the embedding ι'.) Then $\hat{s}_{ij}' = r_i^{-1} \hat{s}_{ij} r_j$. Define $\varphi^f : \Omega^f \to (\Omega')^f$ by

$$\varphi^f(\langle j, y, n, x, i \rangle) = \langle j, y, r_j(y)^{-1} n r_i(x), x, i \rangle.$$

This is well defined because the f_{ij} are central. We verify that $\varphi^f \circ \iota^f = (\iota')^f$. Represent $m \in M$ as $\varrho(\sigma_i(x), n)$, where $\sigma_i = \kappa \circ \hat{\sigma}_i$. Then $(\varphi^f \circ \iota^f)(m) = \langle i, x, r_i(x)^{-1} n r_i(x), x, i \rangle$. To calculate $(\iota')^f(m)$ one must represent m as a value of

$$\kappa' \circ \hat{\sigma}_i' = (\kappa' \circ (\varphi \circ \hat{\sigma}_i)) \cdot (\kappa' \circ r_i) = \sigma_i I_{r_i}.$$

Evidently $m = \varrho((\kappa' \circ \hat{\sigma}_i')(x), r_i(x)^{-1} n r_i(x))$ and one obtains the desired result.

The proof that the action is well defined with respect to representatives of f, and choice of simple open cover, is similar. We thus have a well-defined action of $\check{H}^1(B, ZN)$ on $\mathrm{Opext}(B \times B, M, \langle \Xi, \varrho \rangle)$; the action property itself is evident.

Theorem 3.5. *Let $\langle \Xi, \varrho \rangle$ be a coupling of a pair Lie groupoid $B \times B$ with an LGB M on B for which the obstruction class $e \in \check{H}^2(B, ZN)$ is zero. (Here $N = M_b$ for some $b \in B$.)*

Then the above action of $\check{H}^1(B, ZN)$ on $\mathrm{Opext}(B \times B, M, \langle \Xi, \varrho \rangle)$ *is free and transitive.*

Proof. Consider an operator Lie groupoid

$$\left(M \stackrel{\iota}{\rightarrowtail} \Omega \xrightarrow{(\beta,\alpha)} B \times B, \kappa\right)$$

and a class $f\in \check{H}^1(B,ZN)$, and suppose that

$$\left(M\overset{\iota^f}{\rightarrowtail}\Omega^f\xrightarrow{(\beta^f,\alpha^f)}B\times B,\kappa^f\right)$$

is equivalent to $(M\rightarrowtail\Omega\twoheadrightarrow B\times B,\kappa)$ under $\varphi:\Omega\to\Omega^f$. We must prove that $f=0$.

Take $\{\hat{\sigma}_i:U_i\to\Omega_b\}$ for a simple open cover $\mathscr{U}=\{U_i\}$ of B, as usual, and let $\{\hat{s}_{ij}:U_{ij}\to N\}$ be the transition functions. Choose an index i_0 with

$$b\in U_{i_0}$$

and denote the map $U_i\to(\Omega^f)_b$, $x\mapsto\langle i,x,1,b,i_0\rangle$ by $\hat{\sigma}_i'$. Notice that

$$\hat{\sigma}_{i_0}'(b)=\tilde{b}.$$

Write $(\varphi\circ\hat{\sigma}_i)(x)=\langle i,x,r_i(x),b,i_0\rangle$, thereby defining $r_i:U_i\to N$. Then

$$\varphi(\hat{\sigma}_j(y)\iota(n)\hat{\sigma}_i(x)^{-1})=\langle j,y,r_i(y)nr_i(x)^{-1},x,i\rangle,$$

by calculation. In particular $(\varphi\circ\iota)(m)$, where $m=\varrho(\sigma_i(x),n)$ and $\sigma_i=\kappa\circ\hat{\sigma}_i$, is equal to $\langle i,x,r_i(x)nr_i(x)^{-1},x,i\rangle$. On the other hand, $(\varphi\circ\iota)(m)=\iota^f(m)=\langle i,x,n,x,i\rangle$, where we have used $\kappa'\circ\varphi=\kappa$. Since this holds for all $n\in N$, it follows that each $r_i(x)$ is central.

Now $\hat{\sigma}_j=\hat{\sigma}_i\hat{s}_{ij}$ and $\varphi\circ\hat{s}_{ij}=\hat{s}_{ij}$, so the $\hat{s}_{ij}$ are also the transition functions for the section-atlas $\{\varphi\circ\hat{\sigma}_i:U_i\to(\Omega^f)_b\}$. On the other hand, direct calculation with the formula defining r_i shows that the transition functions for $\{\varphi\circ\hat{\sigma}_i\}$ are $r_i^{-1}\hat{s}_{ij}f_{ij}r_j$. Since r_i and r_j are central, we get $f_{ij}=r_i-r_j$ and this shows that f is the coboundary of $-r\in C^0(\mathscr{U},ZN)$. Thus the action is free.

We sketch the proof that the action is transitive. Consider two operator Lie groupoids, $(M\rightarrowtail\Omega\twoheadrightarrow B\times B,\kappa)$ and $(M'\rightarrowtail\Omega'\twoheadrightarrow B\times B,\kappa')$. Let $\{\sigma_i:U_i\to\Xi_b\}$ be a section-atlas for Ξ with respect to a simple cover $\mathscr{U}$; since each U_i is contractible, and since $\Omega_b(\Xi_b,ZN,\kappa_b)$ is itself a principal bundle, each σ_i can be lifted to $\hat{\sigma}_i:U_i\to\Omega_b$, and

$$\beta_b\circ\hat{\sigma}_i=\mathrm{id}_{U_i}$$

follows. Similarly one obtains a section-atlas $\{\sigma_i':U_i\to\Omega_b'\}$ for Ω'. For the corresponding transition functions we have $I\circ\hat{s}_{ij}=s_{ij}=I\circ\hat{s}_{ij}'$ so the map $f_{ij}=\hat{s}_{ij}^{-1}\hat{s}_{ij}'$ takes values in ZN. Now $\hat{s}_{ij}'=\hat{s}_{ij}f_{ij}=f_{ij}\hat{s}_{ij}$ and a manipulation with these shows that $\{f_{ij}\}$ is a cocycle. $\square$

We submit that Theorems 3.4 and 3.5 give the classification of locally trivial groupoids which is proper to them, as compared with the rather awkward direct translation of the cocycle classification of principal bundles ([17, II§2]).

Results identical in form to Theorems 3.4 and 3.5 hold for any closed crossed module over $B\times B$. See Remark 4.1.

Theorems 3.4 and 3.5 grew out of the author's construction [17] of an integrability obstruction for transitive Lie algebroids on a simply-connected base and we now briefly describe how that obstruction fits into the present framework.

To every principal bundle $P(B,G)$ there is associated the Atiyah sequence [2]

$$\operatorname{ad} P=\frac{P\times\mathfrak{g}}{G}\rightarrowtail\frac{TP}{G}\twoheadrightarrow TB \tag{7}$$

in which $\operatorname{ad} P$ is the Lie algebra bundle associated to $P(B,G)$ through the adjoint representation of G on $\mathfrak{g}$ (and which could be called the gauge algebra bundle) and

$$\frac{TP}{G}$$

is the vector bundle on B obtained by quotienting $TP\to P$ over the action of G. Sections of

$$\frac{TP}{G}$$

are naturally identified with G-invariant vector fields on P and so

$$\Gamma\left(\frac{TP}{G}\right)$$

acquires a Lie bracket. The structure which results was abstracted by Pradines [19b] into the following concept:

Definition 3.6. A *transitive Lie algebroid* on a manifold B is a vector bundle A on B equipped with a surjective morphism $a:A\to TB$ of vector bundles over B, and a bracket $[\ ,\ ]$ on ΓA, the module of global sections of A, which obeys the Jacobi identity, is alternating, and is such that

(i) $a[X,Y]=[aX,aY]$ for all $X,Y\in\Gamma A$;

(ii) $[X,fY]=f[X,Y]+a(X)(f)Y$ for all $X,Y\in\Gamma A$, $f:B\to\mathbb{R}$.

We usually write a transitive Lie algebroid as an exact sequence $L\rightarrowtail A\twoheadrightarrow TB$; note that $[\ ,\ ]$ restricts to ΓL and in fact L is always a Lie algebra bundle ([17, IV§1]).

It was for many years widely believed that every transitive Lie algebroid was the Atiyah sequence of some principal bundle; finally it was announced in [1] by Almeida and Molino that there exist transitive Lie algebroids which are not integrable in this sense (their examples arise from transversally complete foliations). In [17] we constructed a single cohomological invariant attached to a transitive Lie algebroid on a simply-connected base which gives a necessary and sufficient condition for integrability. Namely, given a transitive Lie algebroid $L\rightarrowtail A\twoheadrightarrow TB$ with B simply-connected, let $\mathfrak{g}$ be the fibre-type of L, let $\tilde{G}$ be the simply-connected group corresponding to $\mathfrak{g}$, and let $Z\tilde{G}$ be the centre of $\tilde{G}$. Then the integrability obstruction e of [17] is a certain element of $\check{H}^2(B,Z\tilde{G})$ and A is integrable iff e actually lies in $\check{H}^2(B,D)$ for some discrete subgroup D of $Z\tilde{G}$. (Alternatively one can say that A is integrable iff there exists a connected Lie group G with Lie algebra $\mathfrak{g}$ such that the corresponding element of $\check{H}^2(B, ZG)$ is zero.)

Here $\check{H}^2(B, Z\tilde{G})$ is Čech cohomology with respect to the sheaf of germs of *constant* maps $B \to Z\tilde{G}$ (to denote which we use italic in place of bold for $Z\tilde{G}$). Comparing the construction in [17] with that given here, one can see that the integrability obstruction is in fact the obstruction class for a certain coupling of Lie groupoids which is implicit in the constructions of [17, V§1]. That the integrability obstruction lies in the Čech cohomology with respect to *constant* maps reflects the fact that one is seeking a Lie groupoid (or principal bundle) for which not only the coupling is prescribed, but also the Lie algebroid. The prescription of the Lie algebroid in effect sets the relevant derivatives equal to zero.

Now in Theorem 3.5, $\check{H}^1(B, ZN)$ contains $\check{H}^1(B, ZN)$ and one can easily check that two Lie groupoids $M \rightarrowtail \Omega^\nu \twoheadrightarrow B \times B$, $\nu = 1, 2$ differ by an element of $\check{H}^1(B, ZN)$ iff they have the same coupling *and* the same Lie algebroid. This is a generalisation of the well-known classification of principal bundles with a flat connection by transition functions which are (locally) constant.

4. Three remarks

Remark 4.1. We briefly describe the formulation of these results in terms of principal bundles. Firstly, by an *extension of principal bundles* we understand a sequence (compare [9, 10])

$$N \rightarrowtail Q(B, H) \xrightarrow{\pi} P(B, G)$$

in which π denotes both a surjective morphism of Lie groups $H \twoheadrightarrow G$ such that

$$N \rightarrowtail H \xrightarrow{\pi} G$$

is exact, and a surjective submersion $Q \twoheadrightarrow P$ such that $\pi(\mathrm{id}_B, \pi): Q(B, H) \twoheadrightarrow P(B, G)$ is a morphism of principal bundles. Secondly, by a *closed crossed module of principal bundles* we understand a diagram

$$\begin{tikzcd}
K \arrow[d, tail] & & & \\
N \arrow[d, "\bar{\partial}"'] & & & \varrho: C \to \mathrm{Aut}(N) \\
J \arrow[r, tail] & S(B, C) \arrow[r, two heads, "\chi"] & P(B, G) &
\end{tikzcd}$$

in which the row is to be an extension of principal bundles, and $(N, \partial, C, \varrho)$ is to be a closed crossed module of Lie groups. The definitions of a *coupling of principal bundles*, of the coupling *associated* to an extension of principal bundles, and of a *lifted* or *operator extension of principal bundles*, follow the pattern of Sections 1 and 3.

Theorems 3.4 and 3.5 now take the following form:

Theorem 3.4′. *Let B be a manifold and let G be a Lie group. Let $\langle S(B, \operatorname{Inn} G), \varrho\rangle$ be a coupling of the principal bundle $B(B,1)$ with G. Then there exists a principal bundle $P(B,G)$ such that the coupling associated to the extension $G \rightarrowtail P(B,G) \twoheadrightarrow B(B,1)$ is $\langle S(B, \operatorname{Inn} G), \varrho\rangle$ iff the obstruction class $e \in \check{H}^2(B, ZG)$, defined as in Section 3, is equal to zero.* □

Theorem 3.5′. *With B and G as above, let $\langle S(B, \operatorname{Inn} G), \varrho\rangle$ be a coupling of $B(B,1)$ with G for which the obstruction class $e \in \check{H}^2(B, ZG)$ is zero. Then there is an action of $\check{H}^1(B, ZG)$ on* $\operatorname{Opext}(B(B,1), G, \langle S(B, \operatorname{Inn} G), \varrho\rangle)$*, defined as in Section 3, and it is free and transitive.* □

In this formulation, Theorems 3.4 and 3.5 might appear to be, but in fact are not, restatements of the exactness of a sequence

$$\check{H}^0(B, \operatorname{Inn} G) \to \check{H}^1(B, ZG) \to \check{H}^1(B, G) \to \check{H}^1(B, \operatorname{Inn} G) \to \check{H}^2(B, ZG)$$

obtained from the coefficient sequence $ZG \rightarrowtail G \twoheadrightarrow \operatorname{Inn} G$ (see, for example, [6, §9]). To appreciate the distinction, consider the first map. This takes any global $\varphi : B \to \operatorname{Inn} G$, lifts it over contractible $U_i \subseteq B$ to $\varphi_i : U_i \to G$ and then takes $f_{ij} = \Phi_i^{-1}\Phi_j : U_{ij} \to ZG$. If one now takes Ω as in the preamble to Theorem 3.5 and constructs Ω^f, one sees that Ω and Ω^f are isomorphic as Lie groupoids but — because the Φ_i need not take values in ZG — the groupoids are not equivalent as operator extensions. Thus f does not act as the identity.

Remark 4.2. It needs to be noted that the correspondence between closed crossed modules of principal bundles and of Lie groupoids is bedevilled by the same need for reference-points as is the correspondence between principal bundles and Lie groupoids themselves. Because of this, one must work through the proofs of Theorems 3.4′ and 3.5′, but it will be no surprise that the various needs for reference-points cancel themselves out and that the principal bundle results are identical in form to the Lie groupoid results.

Beyond this, there is a difference in emphasis between the concepts of principal bundle and of Lie groupoid, which affects the way in which one defines equivalence for the two concepts. Consider two extensions of principal bundles

$$N \overset{\iota}{\rightarrowtail} Q(B,H) \overset{\pi}{\twoheadrightarrow} P(B,G), \tag{8}$$

$$N \overset{\iota'}{\rightarrowtail} Q'(B,H') \overset{\pi'}{\twoheadrightarrow} P(B,G), \tag{8$'$}$$

and suppose that $\varphi(\mathrm{id}_B, \varphi): Q(B,H) \to Q'(B,H')$ is an equivalence of principal bundle extensions; that is, $\varphi \circ \iota = \iota'$ and $\pi' \circ \varphi = \pi$. Then the associated extensions of Lie groupoids are

$$\frac{Q \times N}{H} \rightarrowtail \frac{Q \times Q}{H} \twoheadrightarrow \frac{P \times P}{G} \tag{9}$$

and

$$\frac{Q' \times N}{H'} \rightarrowtail \frac{Q' \times Q'}{H'} \twoheadrightarrow \frac{P \times P}{G} \tag{9$'$}$$

(see Section 2) but

$$\tilde{\varphi}: \frac{Q \times Q}{H} \to \frac{Q' \times Q'}{H'}, \qquad \langle v_2, v_1 \rangle \mapsto \langle \varphi(v_2), \varphi(v_1) \rangle,$$

does not induce an identity map

$$\frac{Q \times N}{H} \to \frac{Q' \times N}{H'}.$$

(Note however that an equivalence of Lie groupoid extensions does induce an equivalence of the associated principal bundle extensions.)

This is resolved by considering operator extensions over a fixed coupling. Let (8) and (8)$'$ have the same coupling $(S(B,C), \varrho)$ and assume that $\kappa' \circ \varphi = \kappa$. Because C acts on N, there is an associated LGB

$$\frac{S \times N}{C}$$

and it is easy to see that

$$\frac{Q \times N}{H} \to \frac{S \times N}{C}, \qquad \langle v, n \rangle \mapsto \langle \kappa(v), n \rangle$$

is an isomorphism over B. Similarly

$$\frac{Q' \times N}{C'}$$

is isomorphic to

$$\frac{S \times N}{C}.$$

Now (9) and (9)$'$ can be presented as

$$\frac{S \times N}{C} \rightarrowtail \frac{Q \times Q}{H} \twoheadrightarrow \frac{P \times P}{G}, \tag{10}$$

$$\frac{S \times N}{C} \rightarrowtail \frac{Q' \times Q'}{H'} \twoheadrightarrow \frac{P \times P}{G} \tag{10$'$}$$

and it is easy to check that $\tilde{\varphi}$ is now an equivalence. Thus the two concepts of equivalence are equivalent.

Notice that when $N=A$ is abelian,

$$\frac{S\times N}{C}$$

collapses to

$$\frac{P\times A}{G},$$

where G acts on A via $A \rightarrowtail H \twoheadrightarrow G$.

Remark 4.3. The calculations for Theorems 3.4 and 3.5 are formally identical to those carried out by Greub and Petry [9] in classifying lifts of a principal bundle $P(B,G)$ with G connected, to the universal covering group $\tilde{G}$ (or, indeed, to any cover of G). Their results can in fact be formulated in terms of the closed crossed module

$$\begin{tikzcd} \pi_1 G \arrow[d, tail] & & \\ \tilde{G} \arrow[d] & & \mathrm{Ad}: G \to \mathrm{Aut}(\tilde{G}). \\ G \arrow[r, tail] & P(B,G) \arrow[r, two heads] & B(B,1) \end{tikzcd}$$

Applying the appropriate generalization of Theorems 3.4′ and 3.5′ one recovers the results of [9].

Note that since $\pi_1 G$ is discrete, the Čech cohomology $\check{H}^*(B,\pi_1 G)$ reduces to $\check{H}^*(B,\pi_1 G)$.

Finally, observe that when the integrability obstruction $e\in \check{H}^2(B,Z\tilde{G})$ of a transitive Lie algebroid on a simply-connected base [17] does lie in $\check{H}^2(B,D)$ for some discrete subgroup $D\leq Z\tilde{G}$, then it represents precisely the obstruction to lifting the resulting principal bundle from group G to group $\tilde{G}$.

To end, we mention two examples.

(i) (Greub and Petry [9]). Applying Remark 4.3 with $G=\mathrm{SO}(n)$, $n\geq 3$, gives an obstruction class $e\in \check{H}^2(B,\mathbb{Z}_2)$ to the existence of a covering Spin(n) bundle. This is, of course, the second Stiefel–Whitney class.

(ii) In a similar fashion, consider the relationship between $U(n)$-bundles and $\mathrm{PU}(n)$-bundles. Each principal bundle $P(B,\mathrm{PU}(n))$ defines a coupling

$$\begin{tikzcd}
U(1) \arrow[d, tail] \\
U(n) \arrow[d, two heads] \\
\mathrm{PU}(n) \arrow[r, tail] & P(B, \mathrm{PU}(n)) \arrow[r, two heads] & B(B, 1)
\end{tikzcd}$$

where $\mathrm{PU}(n)$ acts on $U(n)$ as its inner automorphism group. The obstruction class $e \in \check{H}^2(B, U(\mathbf{1}))$ determines whether P can be lifted to a $U(n)$-bundle. Notice that $\check{H}^2(B, U(\mathbf{1})) \cong \check{H}^3(B, \mathbb{Z})$.

This class was studied by Woodward [24], who also introduces it via $\mathbb{Z}_n \rightarrowtail \mathrm{SU}(n) \twoheadrightarrow \mathrm{PU}(n)$ as an element of $\check{H}^2(B, \mathbb{Z}_n)$. It follows that $e \in \check{H}^3(B, \mathbb{Z})$ has $ne = 0$.

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